

# Posterior Consistency via Precision Operators for Bayesian Nonparametric Drift Estimation in SDEs

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## Abstract

We study a Bayesian approach to nonparametric estimation of the periodic drift function of a one-dimensional diffusion from continuous-time data. We rewrite the likelihood in terms of Riemann integrals, by introducing the local time of the process, and specify a centered Gaussian prior on the drift with a precision operator that is of differential form. It is proved that this is a conjugate prior for the likelihood and hence that the posterior is also Gaussian. We give an explicit expression for the posterior precision operator, also of differential form, and show that the posterior mean is the solution of a differential equation requiring inversion of the posterior precision for its solution. Moreover, we bound the rate at which the posterior contracts around the true drift function. Our formulation of the estimation problem leads to algorithms which are readily implementable and analyzed using ideas from the numerical analysis of differential equations. The central results proved here require tools from the analysis of differential equations, together with new functional limit theorems for the local time of diffusions on the circle.

## 1 Introduction

Diffusion processes are routinely used as statistical models for a large variety of phenomena including molecular dynamics, econometrics and climate dynamics (see for instance [30], [21] and [19]). Such a process can be specified via the drift and diffusion functions of a stochastic differential equation driven by a Brownian motion  $W$ . Even in one dimension, this class of processes attracts great applied interest. In this case, provided the diffusion function  $\sigma$  is known and under mild

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additional assumptions, one can transform the process such that the diffusion function is constant:

$$dX_t = b(X_t)dt + dW_t. \quad (1.1)$$

This is the form we consider here.

We are interested in the statistical problem of recovering the drift function  $b$  given an observed path of the diffusion,  $(X_t : t \in [0, T])$ , which is a solution of (1.1). Whenever application-driven insight into the form of the drift  $b$  is available, one can attempt to exploit this by postulating a parametric model for  $b$ , indexed by some finite-dimensional parameter  $\theta \in \Theta \subset \mathbb{R}^d$ . The statistical problem then reduces to estimating the parameter  $\theta$ , see e.g. [22] for an overview of this well-researched area. In other cases however, one has to resort to nonparametric methods for making inference on the function  $b$ . Several such methods have been proposed in the literature. An incomplete list include kernel methods (e.g. [3], [22], [35]), penalized likelihood methods (e.g. [8]), and spectral approaches [2].

In this paper we investigate recently developed Bayesian methodology for estimating the drift function of a diffusion based on continuous-time observations  $X^T = (X_t : t \in [0, T])$ . We consider a periodic setup, which essentially means that we observe a diffusion on the circle. This is motivated by applications in which the data consist of a recording of angles. In molecular dynamics for instance, certain angles in molecules are typical chemically relevant coordinates and can be modeled by diffusions, see [13] and [14]. We will consider Gaussian prior measures for the periodic drift function  $b$  whose inverse covariance operators are chosen from a family of even order differential operators. Recent applied work has shown that this is computationally attractive, since numerical methods for differential equations can be used for posterior sampling. Two illustrative applications of our inference framework and algorithms to accomodate both continuously and discretely observed data are detailed in [26].

In section 2 we precisely state the inference problem of interest, and describe the properties of the family of Gaussian priors that we adopt. We postulate a prior precision operator of the form

$$\mathcal{C}_0^{-1} = \eta ((-\Delta)^p + \kappa I),$$

where  $\Delta$  is the one-dimensional Laplacian,  $p$  is an integer and  $\eta, \kappa$  are real and positive hyperparameters. Working with prior precision operators has numerous computational advantages and a central goal of this work is to develop statistical tools of analysis, in particular for posterior consistency studies, which are well-adapted to this setting. The work of [1] developed tools of analysis which do this in the context of linear inverse problems with small observational noise, and we adapt the techniques developed there to our setting.

An appealing aspect of choosing a Gaussian prior on the drift function  $b$  is conjugacy, in the sense that the posterior is Gaussian as well. Since the log-likelihood is quadratic in  $b$  (Girsanov's theorem) this is not unexpected. Formally the posterior can be computed by "completing the square". We note

however that for our model, if  $b$  is distributed according to a Gaussian prior  $\Pi$  and given  $b$ , the data  $X^T$  are generated by (1.1), the joint distribution of  $b$  and  $X^T$  is obviously *not* Gaussian in general. As a result, deriving Gaussianity of the posterior in this infinite-dimensional setting is not straightforward. Section 3 is devoted to showing that, for the priors that we consider, the posterior, i.e. the conditional distribution of  $b$  given  $X^T$ , is indeed Gaussian. After a formal derivation of the associated posterior in subsection 3.2 we rigorously prove in Theorem 3.2 below that the associated posterior is Gaussian and obtain the posterior mean and covariance structure. The posterior precision operator is again a differential operator, involving the local time of the diffusion, and the posterior mean is characterized as the unique weak solution of a  $2p$ -th order differential equation. In subsection 3.3 we outline how our Bayesian approach with Gaussian prior can be viewed as a penalized least-squares estimator, where the  $p$ th order Sobolev norm of  $b$  is penalized and the hyperparameters  $\eta$  and  $\kappa$  quantify the degree of penalization. In the inverse problem literature this connection is known as Tikhonov-regularisation.

In Bayesian nonparametrics it is well known that careless constructions of priors can lead to inconsistent procedures and sub-optimal convergence rates (e.g. [10], [7]). Consistency or rate of convergence results are often obtained using general results that are available for various types of statistical models and that give sufficient conditions in terms of metric entropy and prior mass assumptions. See, for instance, [17], [15], [16], [33], and the references therein. In this paper however we use the explicit description of the posterior distribution, which allows us to take a rather direct approach to studying the asymptotic behavior of our procedure. In particular, we avoid entropy or prior mass considerations.

Since the posterior involves a periodic version of the local time of the process  $X$ , the asymptotic properties of the local time play a key role in this investigation. In the present setting the existing asymptotic theory for the local time of ergodic diffusions (cf. e.g. [36], [34]) can not be used however, since we do not assume ergodicity but instead rely on the periodicity of the drift function  $b$  to accumulate information as  $T \rightarrow \infty$ . As a consequence, the existing posterior rate of convergence results for ergodic diffusion models of [25] do not apply. In section 4 we therefore present new limit theorems for the local time of diffusions on the circle. These can be seen as extending and complementing the work of Bolthausen [5], who proved a uniform central limit theorem for the local time of Brownian motion on the circle (the case  $b \equiv 0$  in (1.1)). For our purposes we need asymptotic tightness of the properly normalized local time in certain Sobolev spaces however, and we need the result not just for Brownian motion, but for general periodic, zero-mean drift functions  $b$ .

Having these technical tools in place we use them in combination with methods from the analysis of differential equations in section 5 to obtain a rate of contraction result for the posterior distribution. The result states that when the true drift function  $b$  is periodic and  $p$ -regular in the Sobolev sense, then the posterior contracts around  $b$  at a rate that is essentially  $T^{-(p-1/2)/(2p)}$  as  $T \rightarrow \infty$  (with respect to the  $L^2$ -norm). In particular, we have posterior consis-

tency. Although lower bounds for the rate of convergence in the exact model under study do not appear to be known, comparison with similar models suggests that the optimal rate for estimating a drift function  $b_0$  that is  $\beta$ -regular (in Sobolev sense) may be  $T^{-\beta/(1+2\beta)}$  in our setting (in a minimax sense over Sobolev balls for instance, cf. e.g. [22], [32] for similar results). The general message from the Gaussian process prior literature is that this optimal rate is typically attained if the “regularity”  $\alpha$  of the prior matches the regularity  $\beta$  of the function that is being estimated (see [37]). As discussed in Section 2.2, the regularity of the prior we employ in this paper is essentially  $\alpha = p - 1/2$ . This suggests that in principle, it should be possible to relax our assumption that  $b_0$  is  $p$ -regular to the assumption that  $b_0$  is  $(p - 1/2)$ -regular, while still maintaining the same rate  $T^{-(p-1/2)/(2p)}$ . It is however not clear whether this can be achieved by adapting the proof we give in this paper. The method of proof is adapted from [1] where it is used to study linear inverse problems in the small noise limit. In that context the proof gives sharp rates in some parameter regimes, but not in others.

There are a number of future directions that this work could be taken in. First of all, alternative technical approaches could be explored to derive sharp convergence rates. One approach could be to use the representation of the posterior mean as a minimizer of some stochastic objective functional (cf. Section 3.3) and use empirical process-type techniques to study its asymptotic properties. This however requires technical tools (e.g. uniform limit theorems, maximal inequalities) that are presently not available in this setting of periodic diffusions. Alternatively, sharp rates may result from a general rate of convergence theory for posteriors in the spirit of [33], if that could be developed for this class of models. Secondly, motivated by practical considerations, it will also be interesting to determine whether useful adaptive procedures can be constructed by choosing the hyperparameters  $p, \eta$  and  $\kappa$  in a data-driven way, for instance by hierarchical Bayes or empirical Bayes procedures. A third future direction concerns extension of the ideas in this paper to diffusions in more than one dimension. The local time is, then, a much more singular object and developing an analysis of posterior consistency will present new challenges.

## 2 Observation model and prior distribution

In this section we first introduce the diffusion process under study, fixing notation and describing how we exploit periodicity; see subsection 2.1. In Subsection 2.2 we introduce the prior we place on the drift function of the diffusion, specifying the prior precision operator and collecting basic properties.

### 2.1 The diffusion

Consider the stochastic differential equation (SDE)

$$dX_t = b(X_t) dt + dW_t, \quad X_0 = 0, \quad (2.1)$$

where  $W$  is standard Brownian motion and  $b : \mathbb{R} \rightarrow \mathbb{R}$  is a continuously differentiable, 1-periodic drift function with zero mean, i.e.  $b(x+k) = b(x)$  for all  $x \in \mathbb{R}$  and  $k \in \mathbb{Z}$  and  $\int_0^1 b(x)dx = 0$ . We let  $\mathbb{T}$  denote the circle  $[0,1)$  so that we can also write  $b : \mathbb{T} \rightarrow \mathbb{R}$  and we summarize the assumptions on  $b$  by writing  $b \in \dot{C}^1(\mathbb{T})$ , the dot denoting mean zero.

For every  $b \in \dot{C}^1(\mathbb{T})$  the SDE (2.1) has a unique weak solution (see e.g. Theorems 6.1.6 and 6.2.1 in [11], p. 214). For  $T > 0$ , we denote the law that this solution generates on the canonical path space  $C[0, T]$  by  $\mathbb{P}_b^T$ . In particular  $\mathbb{P}_0^T$  is the Wiener measure on  $C[0, T]$ . By Girsanov's theorem the laws  $\mathbb{P}_b^T$ ,  $b \in \dot{C}^1(\mathbb{T})$ , are all equivalent on  $C[0, T]$ . If two measurable maps of  $X$  are almost surely (a.s.) equal under some  $\mathbb{P}_{b_0}^T$  they are therefore a.s. equal under any of the laws  $\mathbb{P}_b^T$ , and we will simply write that they are equal a.s..

Denoting the canonical process on  $C[0, T]$  by  $X$ , the Radon-Nikodym derivative of  $\mathbb{P}_b^T$  relative to the Wiener measure satisfies

$$\frac{d\mathbb{P}_b^T}{d\mathbb{P}_0^T}(X) = \exp\left(-\frac{1}{2} \int_0^T b^2(X_t) dt + \int_0^T b(X_t) dX_t\right)$$

almost surely, by Girsanov's theorem (e.g. [24]). Observe that by Itô's formula the likelihood can be rewritten as

$$\frac{d\mathbb{P}_b^T}{d\mathbb{P}_0^T}(X) = \exp\left(-\Phi_T(b; X)\right)$$

a.s., where

$$\Phi_T(b; X) = \frac{1}{2} \int_0^T (b^2(X_t) + b'(X_t)) dt + B(X_0) - B(X_T) \quad (2.2)$$

and  $B' = b$ . Note that  $B$  is also 1-periodic, since  $b$  has average zero.

It will be convenient to write the integrals in the expression for  $\Phi_T$  in terms of the local time of the process  $X$ . Let  $(L_t(x; X) : t \geq 0, x \in \mathbb{R})$  be the semimartingale local time of  $X$ , so that

$$\int_{-\infty}^{\infty} f(x) L_T(x; X) dx = \int_0^T f(X_s) ds \quad (2.3)$$

holds a.s. for any bounded, measurable  $f : \mathbb{R} \rightarrow \mathbb{R}$ . Defining also the random variables  $\chi_T(x; X)$  by

$$\chi_T(x; X) = \begin{cases} 1 & \text{if } X_0 < x < X_T, \\ -1 & \text{if } X_T < x < X_0, \\ 0 & \text{otherwise,} \end{cases}$$

we may then write

$$\Phi_T(b; X) = \frac{1}{2} \int_{\mathbb{R}} \left( L_T(x; X)(b^2(x) + b'(x)) - 2\chi_T(x; X)b(x) \right) dx. \quad (2.4)$$

In view of the periodicity of the functions involved it is sensible to introduce a periodic version  $L^\circ$  of the local time  $L$  by defining

$$L_T^\circ(x; X) = \sum_{k \in \mathbb{Z}} L_T(x + k; X)$$

for  $x \in \mathbb{T}$ . Note that for every  $T > 0$ , only finitely many term in the sum are almost surely non-zero, so the sum is well defined. It follows from (2.3) that for any 1-periodic, bounded, measurable function  $f$  and  $T \geq 0$ ,

$$\int_0^T f(X_u) du = \int_0^1 f(x) L_T^\circ(x; X) dx. \quad (2.5)$$

Exploiting the periodicity of  $b$  and  $B$  and introducing the corresponding periodized version  $\chi_T^\circ(\cdot; X)$  of  $\chi_T(\cdot; X)$ , we can then rewrite (2.4) as

$$\Phi_T(b; X) = \frac{1}{2} \int_0^1 \left( L_T^\circ(x; X)(b^2(x) + b'(x)) - 2\chi_T^\circ(x; X)b(x) \right) dx. \quad (2.6)$$

We remark that under our assumptions, the random function  $x \mapsto L_T^\circ(x; X)$  is a.s. continuous. This follows from the fact that for the diffusion  $X$  given by (2.1) the local time  $L$  is continuous and there are only finitely non-zero terms in the sum defining  $L^\circ$ , almost surely. In particular, we have that the norms  $\|L_T^\circ(\cdot; X)\|_\infty$  and  $\|L_T^\circ(\cdot; X)\|_{L^2}$  are a.s. finite.

## 2.2 The prior

We will assume that we observe a solution of the SDE (2.1) up to time  $T > 0$ , for some  $b \in \dot{C}^1(\mathbb{T})$ . To make inference on  $b$  we endow it with a centered Gaussian prior  $\Pi$ . We will view the prior as a centered Gaussian measure on  $L^2(\mathbb{T})$  and define it through its covariance operator  $\mathcal{C}_0$ , or, rather, through its precision operator  $\mathcal{C}_0^{-1}$ . Specifically, we fix hyperparameters  $\eta, \kappa > 0$  and  $p \in \{2, 3, \dots\}$  and consider the operator  $\mathcal{C}_0$  with densely defined inverse

$$\mathcal{C}_0^{-1} = \eta ((-\Delta)^p + \kappa I), \quad (2.7)$$

where  $\Delta$  the one-dimensional Laplacian,  $I$  is the identity and the domain of  $\mathcal{C}_0^{-1}$  is given by  $D(\mathcal{C}_0^{-1}) = \dot{H}^{2p}(\mathbb{T})$ , the space of mean-zero functions in the Sobolev space  $H^{2p}(\mathbb{T})$  of functions in  $L^2(\mathbb{T})$  with  $2p$  square integrable weak derivatives.

To see that  $\mathcal{C}_0$  is indeed a valid covariance operator and hence the prior is well defined, consider the orthonormal basis  $\phi_k$  of  $\dot{L}^2(\mathbb{T})$ , which is by definition the space of mean-zero functions in  $L^2(\mathbb{T})$ , given by

$$\begin{aligned} \phi_{2k}(x) &= \sqrt{2} \cos(2\pi kx), \\ \phi_{2k-1}(x) &= \sqrt{2} \sin(2\pi kx), \end{aligned}$$

for  $k \in \mathbb{N}$ . The functions  $\phi_k$  belong to the domain  $\dot{H}^{2p}(\mathbb{T})$  of the operator (2.7) and

$$\begin{aligned}\mathcal{C}_0^{-1}\phi_{2k} &= (\eta(4\pi^2k^2)^p + \eta\kappa)\phi_{2k}, \\ \mathcal{C}_0^{-1}\phi_{2k-1} &= (\eta(4\pi^2k^2)^p + \eta\kappa)\phi_{2k-1},\end{aligned}$$

for  $k \in \mathbb{N}$ . It follows that  $\mathcal{C}_0$  is the operator on  $\dot{L}^2(\mathbb{T})$  which is diagonalized by the basis  $\phi_k$ , with eigenvalues

$$\lambda_k = \left( \eta \left( 4\pi^2 \left\lceil \frac{k}{2} \right\rceil^2 \right)^p + \eta\kappa \right)^{-1}. \quad (2.8)$$

Thus  $\mathcal{C}_0$  is positive definite, symmetric, and trace-class and hence a covariance operator on  $\dot{L}^2(\mathbb{T})$ . It extends to a covariance operator on the whole space  $L^2(\mathbb{T})$  by setting  $\mathcal{C}_0 1 = 0$ .

The integer  $p$  in (2.7) controls the regularity of the prior  $\Pi$  and we assume  $p \geq 2$  to ensure that the drift is  $C^1$  (see below). The parameter  $\eta > 0$  sets an overall scale for the precision. The parameter  $\kappa$  allows us to shift the precisions in every mode by a uniform amount. We employ  $\kappa > 0$  as it simplifies some of the analysis, but  $\kappa = 0$  could be included in the analysis with further work. Likewise we have assumed a mean zero prior, but extensions to include a mean could be made.

The preceding calculations show that the prior  $\Pi$  is the law of the centered Gaussian process  $W = (W(x), x \in \mathbb{T})$  defined by

$$W(x) = \sum_{k \in \mathbb{N}} \sqrt{\lambda_k} \phi_k(x) Z_k, \quad (2.9)$$

for  $Z_1, Z_2, \dots$  independent, standard Gaussian random variables. Note that  $\sqrt{\lambda_k} \sim k^{-p}$  asymptotically. Using also the differential relations between the basis functions  $\phi_k$  it is straightforward to see that the process  $W$  has  $p - 1 \geq 1$  weak derivatives in the  $L^2$ -sense. Moreover, using Kolmogorov's classical continuity theorem it can be shown that this  $(p - 1)$ st derivative has a version with sample paths that are Hölder continuous of order  $\gamma$  for every  $\gamma < 1/2$ . Combining this we see that  $W$  has a version with  $\alpha$ -Hölder sample paths, for every  $\alpha < p - 1/2$ . In particular, it holds that all the mass of the prior  $\Pi$  is concentrated on  $\dot{C}^1(\mathbb{T})$ .

Note that the Karhunen-Loève expansion (2.9) shows that the reproducing kernel Hilbert space (RKHS) of the prior is given by  $\mathbb{H} = \{\sum_{k \geq 1} c_k \phi_k : \sum c_k^2 / \lambda_k < \infty\}$ . Since  $1/\lambda_k \sim k^{2p}$ , this implies that  $\mathbb{H} = \dot{H}^p(\mathbb{T})$ . This shows once again that the  $L^2$ -support of  $\Pi$  is  $\dot{L}^2(\mathbb{T})$ . We can thus view  $\Pi$  as a Gaussian measure on any of the separable Banach spaces  $L^2(\mathbb{T})$ ,  $C(\mathbb{T})$ ,  $C^k(\mathbb{T})$  or  $H^k(\mathbb{T})$ , for  $k \leq p - 1$ .

### 3 Posterior distribution

#### 3.1 Bayes' formula

If we endow  $C^1(\mathbb{T})$  with its Hölder norm and  $C[0, T]$  with the uniform norm, then expression (2.2) shows that the negative log-likelihood  $(b, x) \mapsto \Phi_T(b; x)$  (has a version that) is Borel-measurable as a map from  $C^1(\mathbb{T}) \times C[0, T] \rightarrow \mathbb{R}$ . Since we can view  $\Pi$  as a measure on  $C^1(\mathbb{T})$ , it follows that we have a well-defined Borel measure  $\Pi(db)\mathbb{P}_b^T(dx)$  on  $C^1(\mathbb{T}) \times C[0, T]$ , which is the joint law of  $b$  and  $X^T$  in the Bayesian setup

$$b \sim \Pi, \quad X^T | b \sim (2.1).$$

The posterior distribution, i.e. the conditional distribution of  $b$  given  $X^T$ , is then well-defined as well and given by

$$\begin{aligned} \Pi(B | X^T) &= \frac{1}{Z} \int_B \exp(-\Phi_T(b; X)) \Pi(db), \\ Z &= \int_{C^1(\mathbb{T})} \exp(-\Phi_T(b; X)) \Pi(db), \end{aligned} \tag{3.1}$$

for Borel sets  $B \subset C^1(\mathbb{T})$ , provided that  $Z > 0$  a.s., cf. Lemma 5.3 of [18]. To see that the latter condition is fulfilled, observe that since  $\Pi$  is a centered Gaussian distribution on the separable Banach space  $C^1(\mathbb{T})$ , endowed with its Hölder norm, we have  $\Pi(b : \|b\|_\infty + \|b'\|_\infty < \infty) = 1$ . It follows from (2.6) that

$$|\Phi_T(b; X)| \lesssim (1 + \|L_T^\circ(\cdot; X)\|_\infty)(\|b\|_\infty^2 + \|b'\|_\infty).$$

(Here, and elsewhere,  $a \lesssim b$  means that  $a$  is less than an irrelevant constant times  $b$ .) Together this gives the a.s. positivity of  $Z$ , since  $\|L_T^\circ(\cdot; X)\|_\infty < \infty$  a.s.

We have now defined the posterior as a measure on  $C^1(\mathbb{T})$ , but since the prior is in fact a probability measure on  $C^\alpha(\mathbb{T})$  for every  $\alpha < p - 1/2$  (see the preceding section), it is a Borel measure on these Hölder spaces as well. We can of course also view it as a measure on  $C(\mathbb{T})$  or  $L^2(\mathbb{T})$ .

#### 3.2 Formal computation of the posterior

The next goal is to characterize the posterior. We proceed first strictly formally and non-rigorously. Very loosely speaking, we have that the prior  $\Pi$  has a “density” proportional to

$$b \mapsto \exp\left(-\frac{1}{2} \int_0^1 b(x) C_0^{-1} b(x) dx\right) \tag{3.2}$$

and the negative log-likelihood also has a quadratic form, given by (2.6). This suggests that the posterior is again Gaussian. Formally completing the square



gives the relations

$$\mathcal{C}_T^{-1} = \mathcal{C}_0^{-1} + L_T^\circ(\cdot; X)I, \quad (3.3)$$

$$\mathcal{C}_T^{-1}\hat{b}_T = \frac{1}{2}(L_T^\circ(\cdot; X))' + \chi_T^\circ(\cdot; X) \quad (3.4)$$

for the posterior mean  $\hat{b}_T$  and the posterior precision operator  $\mathcal{C}_T^{-1}$ .

As detailed in the preceding section we assume that the prior covariance operator is given by (2.7), with integer  $p \geq 2$ ,  $\eta, \kappa > 0$ ,  $\Delta$  the one-dimensional Laplacian and  $D(\mathcal{C}_0^{-1}) = \dot{H}^{2p}(\mathbb{T})$ . In that case (3.3) gives

$$\mathcal{C}_T^{-1} = \eta(-\Delta)^p + (\eta\kappa + L_T^\circ(\cdot; X))I \quad (3.5)$$

and  $D(\mathcal{C}_T^{-1}) = \dot{H}^{2p}(\mathbb{T})$ . By standard PDE theory, the equation  $\mathcal{C}_T^{-1}f = g$  has a unique weak solution in  $\dot{H}^{2p}(\mathbb{T})$  for every  $g \in \dot{L}^2(\mathbb{T})$ , see e.g. [12] or [28] for more on the periodic case, hence  $\mathcal{C}_T$  is well defined on all of  $\dot{L}^2(\mathbb{T})$ . Moreover,  $\mathcal{C}_T$  is a bounded operator, since  $\mathcal{C}_T^{-1}$  is coercive.

Of course the ordinary derivative of local time is not defined and we have to interpret (3.4) in a weak sense. In order to enable us to do this, in Section 3.4 we consider the variational formulation of equation (3.4). As a precursor to this, the next subsection is devoted to observing that the differential equation for the mean arises as the Euler-Lagrange equation for a certain variational problem, yielding an interesting connection with penalized least-squares estimation.

### 3.3 Connection with penalized least squares

Here we demonstrate the fact that the posterior mean  $\hat{b}_T$  given by (3.4) can be viewed as a penalized least-squares estimator in the case  $p = 2$ . Formally, the SDE (2.1) can be written as

$$\dot{X}_t = b(X_t) + \dot{W}_t,$$

where the dot denotes differentiation with respect to  $t$  (obviously, the derivatives  $\dot{X}$  and  $\dot{W}$  do not exist in the ordinary sense.) This is just a continuous-time version of a standard nonparametric regression model and for a drift function  $u$ , we can view the integral

$$\int_0^T (\dot{X}_t - u(X_t))^2 dt$$

as a residual sum of squares. A penalized least-squares procedure consists in adding a penalty term to this quantity and minimizing the resulting criterion over  $u$ . Expanding the square in the preceding integral shows that this is equivalent to minimizing

$$u \mapsto - \int_0^T u(X_t) dX_t + \frac{1}{2} \int_0^T u^2(X_t) dt + P(u),$$

over an appropriate space of functions, where  $P(u)$  is the penalty.

If the function  $u$  is smooth and periodic, then by Itô's formula and the definitions of  $L^\circ$  and  $\chi^\circ$ , we have

$$\begin{aligned} \int_0^T u(X_t) dX_t &= U(X_T) - U(X_0) - \frac{1}{2} \int_0^T u'(X_t) dt \\ &= \int_0^1 u(x) \chi_T^\circ(x; X) dx - \frac{1}{2} \int_0^1 L_T^\circ(x; X) u'(x) dx \end{aligned}$$

and

$$\int_0^T u^2(X_t) dt = \int_0^1 u^2(x) L_T^\circ(x; X) dx.$$

Hence, if the functions  $u$  over which the minimization takes place are smooth enough, the criterion can also be written as

$$u \mapsto \int_0^1 \left( \frac{1}{2} u^2(x) L_T^\circ(x; X) + \frac{1}{2} u'(x) L_T^\circ(x; X) - u(x) \chi_T^\circ(x; X) \right) dx + P(u).$$

Now consider a Sobolev-type penalty term of the form

$$P(u) = \frac{1}{2} \eta \left( \kappa \int_0^1 (u(x))^2 dx + \int_0^1 (u''(x))^2 dx \right),$$

for constants  $\eta, \kappa > 0$ . Then the objective functional  $u \mapsto \Lambda(u; X)$  takes the form

$$\Lambda(u; X) = \int_0^1 \left( \frac{1}{2} u^2(\eta\kappa + L_T^\circ(X)) + \frac{1}{2} u' L_T^\circ(X) - u \chi_T^\circ(X) + \frac{1}{2} \eta (u'')^2 \right) dx,$$

where we omitted explicit dependence on  $x$  to lighten notation. To maximise this functional, simply take its variational derivative in the direction  $v$ , i.e. compute the limit  $\lim_{\epsilon \rightarrow 0} (\Lambda(u + \epsilon v; X) - \Lambda(u; X)) / \epsilon$ , for a smooth test function  $v$ :

$$\frac{\delta \Lambda}{\delta u}(v) = \int_0^1 \left( uv L_T^\circ(X) - \frac{1}{2} v (L_T^\circ)'(X) - v \chi_T^\circ(X) + \eta v'' u'' + \eta \kappa uv \right) dx$$

Integrating by parts (where the boundary terms vanish due to periodicity) now yields the form

$$\frac{\delta \Lambda}{\delta u}(v) = \int_0^1 v \left( u L_T^\circ(X) - \frac{1}{2} (L_T^\circ)'(X) - \chi_T^\circ(X) + \eta u^{(4)} + \eta \kappa u \right) dx$$

from which it is evident that equating the variational derivative to zero for all smooth test functions yields exactly the posterior mean obtained in (3.4) for the case  $p = 2$ :

$$\eta u^{(4)} + (\eta \kappa + L_T^\circ(X)) u = \frac{1}{2} (L_T^\circ)'(X) + \chi_T^\circ(X).$$

In the context of inverse problems, adding the square of the norm of the underlying vector space is known as (generalized) Tikhonov regularization, and the connection to Bayesian inference with a Gaussian prior is well established in general, see [31]. It may be viewed as a natural extension of the approach of Wahba [38] from regression to the diffusion process setting. The case of regularization through higher order derivatives in the penalization term  $P$  is similar.

### 3.4 Weak variational formulation for the posterior mean

In the preceding section we remarked that the RKHS of the Gaussian prior equals the Sobolev space  $\dot{H}^p(\mathbb{T})$ . Below we prove that the posterior is a.s. a Gaussian measure. Moreover, since the denominator  $Z$  in (3.1) is positive a.s., the posterior is equivalent to the prior. It follows that the posterior mean  $\hat{b}_T$  is a.s. an element of  $\dot{H}^p(\mathbb{T})$ . By saying it is a weak solution to (3.4) we mean that it solves the following weak form of the associated variational principle:

$$a(\hat{b}_T, v; X) = r(v; X) \quad \text{for every } v \in \dot{H}^p(\mathbb{T}), \quad (3.6)$$

where the bilinear form  $a(\cdot, \cdot; X) : \dot{H}^p(\mathbb{T}) \times \dot{H}^p(\mathbb{T}) \rightarrow \mathbb{R}$  and the linear form  $r(\cdot; X) : \dot{H}^p(\mathbb{T}) \rightarrow \mathbb{R}$  are defined by

$$\begin{aligned} a(u, v; X) &= \eta \int u^{(p)}(x) v^{(p)}(x) dx + \eta \kappa \int u(x) v(x) dx + \int u(x) v(x) L_T^\circ(x; X) dx, \\ r(v; X) &= -\frac{1}{2} \int v'(x) L_T^\circ(x; X) dx + \int v(x) \chi_T^\circ(x; X) dx. \end{aligned}$$

The following lemma records the essential properties of  $a$  and  $r$  and the associated variational problem.

**Lemma 3.1.** *The following statements hold almost surely:*

(i)  $a(\cdot, \cdot; X)$  is bilinear, symmetric, and continuous and coercive:

$$a(v_1, v_2; X) \leq (\eta + \eta \kappa + \|L_T^\circ(\cdot; X)\|_\infty) \|v_1\|_{H^p} \|v_2\|_{H^p}$$

for  $v_1, v_2 \in \dot{H}^p(\mathbb{T})$  and for some constant  $c > 0$ ,  $a(v, v; X) \geq c \|v\|_{H^p}^2$  for all  $v \in \dot{H}^p(\mathbb{T})$ .

(ii)  $r(\cdot; X)$  is linear and bounded:

$$|r(v; X)| \leq \left( \frac{1}{2} \|L_T^\circ(\cdot; X)\|_{L^2} + \|\chi_T^\circ(\cdot; X)\|_{L^2} \right) \|v\|_{H^p}$$

for all  $v \in \dot{H}^p(\mathbb{T})$ .

(iii) There exists a unique  $u \in \dot{H}^p(\mathbb{T})$  such that  $a(u, v; X) = r(v; X)$  for all  $v \in \dot{H}^p(\mathbb{T})$ .

**Proof.** (i) Bilinearity, symmetry and continuity follow straightforwardly from the definition of  $a$ . Coercivity follows easily from the positivity of  $\eta$  and  $\kappa$  and the Poincaré inequality. (ii) Again, straightforward. (iii) Follows from (i) and (ii) by the Lax-Milgram Lemma, see [6].  $\square$

### 3.5 Characterization of the posterior

We can now prove that the posterior is Gaussian and characterize its mean and covariance operator. Recall that by saying that  $\hat{b}_T$  is a weak solution of the differential equation (3.4) we mean that it solves the variational problem (3.6).

**Theorem 3.2.** *Almost surely, the posterior  $\Pi(\cdot | X^T)$  is a Gaussian measure on  $L^2(\mathbb{T})$ . Its covariance operator  $\mathcal{C}_T$  is given by (3.5) and its mean  $\hat{b}_T$  is the unique weak solution of (3.4).*

**Proof.** For  $n \in \mathbb{N}$ , let  $P_n : L^2(\mathbb{T}) \rightarrow L^2(\mathbb{T})$  be the orthogonal projection onto the linear span  $V_n$  of the first  $n$  basis functions  $\phi_1, \dots, \phi_n$ . Let the random measure  $\Pi_n(\cdot | X^T)$  be given by

$$\begin{aligned}\Pi_n(B | X^T) &= \frac{1}{Z_n} \int_B \exp(-\Phi_T(P_n b; X)) \Pi(db), \\ Z_n &= \int_{C^1(\mathbb{T})} \exp(-\Phi_T(P_n b; X)) \Pi(db),\end{aligned}$$

for Borel sets  $B \subset C^1(\mathbb{T})$ . The fact that this random measure is well defined can be argued exactly as in Section 3.1.

For  $b \in \dot{C}^1(\mathbb{T})$  it holds that  $P_n b \rightarrow b$  in  $H^1(\mathbb{T})$  as  $n \rightarrow \infty$ . It is easily seen from (2.6) that the random map  $b \mapsto \Phi_T(b; X)$  is a.s.  $H^1(\mathbb{T})$ -continuous. It follows that a.s.,  $b \mapsto \Phi_T(P_n b; X)$  converges pointwise to  $\Phi_T(\cdot; X)$  on  $\dot{C}^1(\mathbb{T})$ . By Lemma 3.3 below, there exists for every  $\varepsilon > 0$  a constant  $K(\varepsilon)$  such that

$$-\Phi_n(b; X) \leq \varepsilon \|b\|_{H^1}^2 + K(\varepsilon)(1 + \|L_T^\circ(\cdot; X)\|_{L^2}^2),$$

and hence

$$e^{-\Phi_n(P_n b; X)} \leq e^{K(\varepsilon)(1 + \|L_T^\circ(\cdot; X)\|_{L^2}^2)} e^{\varepsilon \|b\|_{H^1}^2}.$$

Since  $\Pi$  can be viewed as a Gaussian measure on  $H^1(\mathbb{T})$ , Fernique's theorem implies that a.s., the right-hand side of the last display is a  $\Pi$ -integrable function of  $b$  for  $\varepsilon > 0$  small enough (see [4], Theorem 2.8.5). Hence, by dominated convergence, we can conclude that  $Z_n \rightarrow Z$  almost surely. The same reasoning shows that for every Borel set  $B \subset C^1(\mathbb{T})$ , it a.s. holds that

$$\int_B \exp(-\Phi_T(P_n b; X)) \Pi(db) \rightarrow \int_B \exp(-\Phi_T(b; X)) \Pi(db)$$

as  $n \rightarrow \infty$ . Hence, we have that with probability 1, the measures  $\Pi_n(\cdot | X^T)$  converge weakly to the posterior  $\Pi(\cdot | X^T)$ . Note that the weak convergence

takes place in  $C^1(\mathbb{T})$ , but then in  $L^2(\mathbb{T})$  as well. Since the measures  $\Pi_n(\cdot | X^T)$  are easily seen to be Gaussian, the measure  $\Pi(\cdot | X^T)$  must be Gaussian as well.

If we view  $\dot{L}^2(\mathbb{T})$  as the product of  $V_n$  and  $V_n^\perp$ , then by construction the measure  $\Pi_n(\cdot | X^T)$  is a product of Gaussian measures on  $V_n$  and  $V_n^\perp$ . The measure on  $V_n$  really has density proportional (3.2) on  $V_n$  (relative to the pushforward measure of the Lebesgue measure on  $\mathbb{R}^n$  under the map  $(c_1, \dots, c_n) \mapsto \sum c_k \phi_k$ ). The formal arguments given in Section 3.2 can therefore be made rigorous, showing that this factor is a Gaussian measure on  $V_n$  with covariance operator  $P_n \mathcal{C}_T P_n$  and mean  $b_n \in V_n$  which solves the variational problem

$$a(b_n, v; X) = r(v; X)$$

for every  $v \in V_n$ . The measure on  $V_n^\perp$  has mean zero, so  $b_n$  is in fact the mean of the whole measure  $\Pi_n(\cdot | X^T)$ . The covariance operator of the measure on  $V_n^\perp$  is given by  $(I - P_n) \mathcal{C}_0 (I - P_n)$ .

Next we prove that the posterior mean  $\hat{b}_T$  is the weak solution of (3.4). By Lemma 3.1 there a.s. exists a unique  $u \in \dot{H}^p(\mathbb{T})$  such that  $a(u, v; X) = r(v; X)$  for all  $v \in \dot{H}^p(\mathbb{T})$ . Standard Galerkin method arguments show that for the mean of  $\Pi_n(\cdot | X^T)$  we have  $b_n \rightarrow u$  in  $\dot{H}^p(\mathbb{T})$ . Indeed, let  $e_n = u - b_n$ . Then we have the orthogonality property  $a(e_n, v; X) = 0$  for all  $v \in V_n$ . Using the continuity and coercivity of  $a(\cdot, \cdot; X)$ , cf. Lemma 3.1, it follows that for  $v \in V_n$ ,

$$\begin{aligned} c \|e_n\|_{\dot{H}^p}^2 &\leq a(e_n, e_n; X) \\ &= a(e_n, u - v; X) \\ &\leq (\eta + \eta\kappa + \|L_T^\circ(\cdot; X)\|_\infty) \|e_n\|_{\dot{H}^p} \|u - v\|_{\dot{H}^p}. \end{aligned}$$

Hence, for every  $v \in V_n$  we have

$$c \|e_n\|_{\dot{H}^p} \leq (\eta + \eta\kappa + \|L_T^\circ(\cdot; X)\|_\infty) \|u - v\|_{\dot{H}^p}.$$

By taking  $v = P_n u$  we then see that  $b_n \rightarrow u$  in  $\dot{H}^p(\mathbb{T})$ . On the other hand, by the weak convergence found above,  $b_n$  converges a.s. to the posterior mean  $\hat{b}_T$  in  $L^2(\mathbb{T})$  (see [4], Example 3.8.15). We conclude that  $\hat{b}_T$  a.s. equals the unique weak solution  $u$  of (3.4), as required.

It remains to show that the covariance operator of the posterior is given by (3.5). Let  $\Sigma_n = P_n \mathcal{C}_T P_n + (I - P_n) \mathcal{C}_0 (I - P_n)$  be the covariance operator of  $\Pi_n(\cdot | X^T)$  and let  $\Sigma$  be the covariance operator of the posterior  $\Pi(\cdot | X^T)$ . Since the measures converge weakly and are Gaussian, we have that for every  $f \in L^2(\mathbb{T})$ ,  $\Sigma_n f \rightarrow \Sigma f$  in  $L^2(\mathbb{T})$  (cf. Example 3.8.15 of [4] again). On the other hand, for  $n > k$  and  $g \in \dot{L}^2(\mathbb{T})$  we have

$$\begin{aligned} \left| \langle g, \Sigma_n \phi_k \rangle - \langle g, \mathcal{C}_T \phi_k \rangle_{L^2} \right| &= \left| \langle g, (P_n - I) \mathcal{C}_T \phi_k \rangle_{L^2} \right| \\ &\leq \|(P_n - I)g\|_{L^2} \|\mathcal{C}_T \phi_k\|_{L^2} \rightarrow 0, \end{aligned}$$

hence  $\Sigma_n \phi_k$  converges weakly to  $\mathcal{C}_T \phi_k$ . It follows that  $\Sigma \phi_k = \mathcal{C}_T \phi_k$  for every  $k$  and the proof is complete.  $\square$

**Lemma 3.3.** *For every  $\varepsilon > 0$  there exists a constant  $K(\varepsilon) > 0$  such that*

$$-\Phi(b; X) \leq \varepsilon \|b\|_{H^1}^2 + K(\varepsilon)(1 + \|L_T^\circ(\cdot; X)\|_{L^2}^2).$$

**Proof.** It follows from (2.6) that

$$-\Phi(b; X) \leq +\frac{1}{2} \int_0^1 L_T^\circ(x; X) |b'(x)| dx + \int_0^1 |b(x)| dx.$$

Now note that for every  $\beta > 0$  and  $f, g \in L^2(\mathbb{T})$ , it holds that  $2\langle f, g \rangle \leq \beta \|f\|^2 + \beta^{-1} \|g\|^2$  (“Young’s inequality with  $\varepsilon$ ”). Applying this to both integrals on the right we get

$$\begin{aligned} -\Phi(b; X) &\leq \frac{\beta}{4} \|L_T^\circ(\cdot; X)\|_{L^2}^2 + \frac{1}{4\beta} \|b'\|_{L^2}^2 + \frac{\beta}{2} + \frac{1}{2\beta} \|b\|_{L^2}^2 \\ &\leq \varepsilon \|b\|_{H^1}^2 + \frac{1}{4\varepsilon} + \frac{1}{8\varepsilon} \|L_T^\circ(\cdot; X)\|_{L^2}^2, \end{aligned}$$

where  $\varepsilon = (2\beta)^{-1}$ , so  $K(\varepsilon) = \frac{1}{4\varepsilon}$ . □

## 4 Asymptotic behaviour of the local time

In the next section we will investigate the asymptotic behaviour of the posterior, using the characterization provided by Theorem 3.2. Since (3.3) and (3.4) involve the periodic local time  $L_T^\circ(\cdot; X)$ , the asymptotic properties of that random function play a key role.

The results we establish in this section can be seen as complementing and extending the work of Bolthausen [5] in which it is proved that if  $X$  is Brownian motion (i.e.  $b \equiv 0$  in (2.1)), then the random functions

$$\sqrt{T} \left( \frac{1}{T} L_T^\circ(\cdot; X) - 1 \right)$$

converge weakly in the space  $C(\mathbb{T})$  to a Gaussian random map as  $T \rightarrow \infty$ . For our purposes we need weak convergence (or at least asymptotic tightness) in the Sobolev space  $H^\alpha(\mathbb{T})$ , for  $\alpha < 1/2$ , and we need the result not just for Brownian motion, but for general periodic, zero-mean drift functions  $b$ . Moreover, we need the associated uniform law of large numbers which states that

$$\frac{1}{T} L_T^\circ(\cdot; X)$$

converges uniformly as  $T \rightarrow \infty$ . Similar statements were obtained for ergodic diffusions in the papers [36] and [34]. In the present periodic setting however, completely different arguments are necessary.

Given  $b \in \dot{C}(\mathbb{T})$  we define the probability density  $\rho$  on  $[0, 1]$  by

$$\rho(x) = C \exp \left( 2 \int_0^x b(y) dy \right), \quad x \in [0, 1], \quad (4.1)$$

where  $C > 0$  is the normalization constant that ensures that  $\rho$  integrates to 1. Note that since  $b$  has mean zero,  $\rho$  satisfies  $\rho(0) = \rho(1)$  and enjoys a natural extension to a periodic function.

**Theorem 4.1.**

(i) *It almost surely holds that*

$$\sup_{x \in \mathbb{T}} \left| \frac{1}{T} L_T^\circ(x; X) - \rho(x) \right| \rightarrow 0$$

as  $T \rightarrow \infty$ .

(ii) *For every  $\alpha < 1/2$ , the random maps*

$$x \mapsto \sqrt{T} \left( \frac{1}{T} L_T^\circ(x; X) - \rho(x) \right)$$

*are asymptotically tight in  $H^\alpha(\mathbb{T})$  as  $T \rightarrow \infty$ . In particular, for every  $\alpha < 1/2$ ,*

$$\left\| \frac{1}{T} L_T^\circ(\cdot; X) - \rho \right\|_{H^\alpha} = O_P\left(\frac{1}{\sqrt{T}}\right)$$

as  $T \rightarrow \infty$ .

The proof of the theorem is long and therefore deferred to the final section 6 in order to keep the overarching arguments in this paper, which are aimed to proving posterior consistency, to the fore. In the following section about posterior contraction rates we need the fact that  $X_T = O_P(\sqrt{T})$  for the diffusions with periodic drift under consideration. As we are not aware of an existing reference to this statement, we derive it from the preceding theorem.

**Corollary 4.2.** *For every  $b \in \dot{C}(\mathbb{T})$ , the weak solution  $X = (X_t : t \geq 0)$  of the SDE (2.1) satisfies  $X_T = O_P(\sqrt{T})$  as  $T \rightarrow \infty$ .*

**Proof.** We have

$$X_T = \int_0^T b(X_s) ds + W_T$$

for a standard Brownian motion  $W$ . Since  $b$  is 1-periodic, the integral can be rewritten in terms of the periodic local time  $L^\circ$ . Moreover, (4.1) implies that  $\rho$  is 1-periodic as well and  $\rho' = 2b\rho$ , hence

$$\int_0^1 b(x) \rho(x) dx = \frac{1}{2}(\rho(1) - \rho(0)) = 0.$$

It follows that

$$\begin{aligned} |X_T| &\leq T \left| \int_0^1 b(x) \left( \frac{L_T^\circ(x; X)}{T} - \rho(x) \right) dx \right| + |W_T| \\ &\leq T \|b\|_{L^2} \left\| \frac{1}{T} L_T^\circ(\cdot; X) - \rho \right\|_{L^2} + |W_T|. \end{aligned}$$

By the preceding theorem, this is  $O_P(\sqrt{T})$ . □

## 5 Posterior contraction rates

In this section we use the characterization of the posterior provided by Theorem 3.2 and the asymptotic behaviour of the local time established in Theorem 4.1 to study the rate at which the posterior contracts around the true drift function, which we denote by  $b_0$  to emphasize that the results are frequentist in nature.

The first theorem concerns the rate of convergence of the posterior mean  $\hat{b}_T$ , which, by Theorem 3.2, is the unique weak solution in  $\dot{H}^p(\mathbb{T})$  of the differential equation (3.4).

**Theorem 5.1.** *Suppose that the true drift function  $b_0 \in \dot{H}^p(\mathbb{T})$ . Then for every  $\delta > 0$ ,*

$$\|\hat{b}_T - b_0\|_{L^2} = O_{\mathbb{P}_{b_0}}(T^{-\frac{p-1/2}{2p}+\delta})$$

as  $T \rightarrow \infty$ .

**Proof.** By Theorem 3.2 we have (in weak sense)

$$(\mathcal{C}_0^{-1} + L_T^\circ(\cdot; X))\hat{b}_T = \frac{1}{2}(L_T^\circ(\cdot; X))' + \chi_T^\circ(\cdot; X).$$

Note that it follows from (4.1) that the invariant density  $\rho$  satisfies  $\rho' = 2b_0\rho$  if  $b_0$  is the drift function, hence, with  $\mathbb{G}_T = \sqrt{T}(L_T^\circ(\cdot; X)/T - \rho)$ ,

$$\begin{aligned} (\mathcal{C}_0^{-1} + L_T^\circ(\cdot; X))b_0 &= \frac{1}{2}(L_T^\circ(\cdot; X))' + \chi_T^\circ(\cdot; X) + \mathcal{C}_0^{-1}b_0 \\ &\quad + \sqrt{T}\mathbb{G}_T b_0 - \frac{1}{2}\sqrt{T}\mathbb{G}_T' - \chi_T^\circ(\cdot; X). \end{aligned}$$

Subtracting the two equations shows that  $e = \hat{b}_T - b_0$  satisfies (still in the weak sense)

$$(\mathcal{C}_0^{-1} + L_T^\circ(\cdot; X))e = -\mathcal{C}_0^{-1}b_0 - \sqrt{T}\mathbb{G}_T b_0 + \frac{1}{2}\sqrt{T}\mathbb{G}_T' + \chi_T^\circ(\cdot; X).$$

Since  $\rho$  is bounded away from zero (see (4.1)) it follows from statement (i) of Theorem 4.1 that there exists a constant  $c > 0$  such that

$$\inf_{x \in \mathbb{T}} \frac{L_T^\circ(x; X)}{T} \geq c$$

on an event  $A_T$  with  $\mathbb{P}_{b_0}(A_T) \rightarrow 1$ . As a consequence, testing the weak differential equation for the error  $e$  with the test function  $e$  itself (“energy method”) yields, on the event  $A_T$ , the inequality

$$\begin{aligned} &\|\mathcal{C}_0^{-1/2}e\|_{L^2}^2 + cT\|e\|_{L^2}^2 \\ &\leq |\langle \mathcal{C}_0^{-1}b_0, e \rangle| + |\langle \chi_T^\circ(\cdot; X), e \rangle| + \sqrt{T}|\langle \mathbb{G}_T b_0, e \rangle| + \frac{1}{2}\sqrt{T}|\langle \mathbb{G}_T, e' \rangle| \\ &= |\langle \mathcal{C}_0^{-\frac{1}{2}}b_0, \mathcal{C}_0^{-\frac{1}{2}}e \rangle| + |\langle \chi_T^\circ(\cdot; X), e \rangle| + \sqrt{T}|\langle \mathbb{G}_T b_0, e \rangle| + \frac{1}{2}\sqrt{T}|\langle \mathbb{G}_T, e' \rangle|. \end{aligned}$$



Using Young's inequality  $2\langle f, g \rangle \leq \kappa \|f\|_{L^2}^2 + \kappa^{-1} \|g\|_{L^2}^2$  on the first three terms on the right with appropriate  $\kappa$ 's and subtracting the resulting terms involving  $T\|e\|_{L^2}^2$  from both sides we get, still on  $A_T$ ,

$$\begin{aligned} & \|C_0^{-1/2}e\|_{L^2}^2 + cT\|e\|_{L^2}^2 \\ & \lesssim \|C_0^{-1/2}b_0\|_{L^2}^2 + \frac{\|\chi_T^\circ(\cdot; X)\|_{L^2}^2}{T} + \|b_0\|_\infty^2 \|\mathbb{G}_T\|_{L^2}^2 + \frac{1}{2}\sqrt{T} |\langle \mathbb{G}_T, e' \rangle|. \end{aligned} \quad (5.1)$$

We note that first three terms on the right are now stochastically bounded: the first one is constant, the second is bounded by a  $|X_T - X_0|^2/T$ , which is  $O_P(1)$  according to Corollary 4.2, and the third one is  $O_P(1)$  by Theorem 4.1.

For the last term on the right we have, since the norm  $\|C^{-s/(2p)} \cdot\|_{L^2}$  is equivalent to the  $H^s$ -norm and  $C_0^{1/(2p)} \frac{\partial}{\partial x}$  is bounded,

$$|\langle \mathbb{G}_T, e' \rangle| = \left| \left\langle C_0^{-\frac{s}{2p}} \mathbb{G}_T, C_0^{\frac{s}{2p}} e' \right\rangle \right| \lesssim \|\mathbb{G}_T\|_{H^s} \|C_0^{\frac{s-1}{2p}} e\|_{L^2}.$$

We now use the interpolation inequality given as Theorem 13 on p.149 in [9],

$$\|A^\iota u\| \leq \|Au\|^\iota \|u\|^{1-\iota},$$

which is valid for  $\iota \in (0, 1)$  and positive, coercive, self-adjoint densely defined operators  $A$ . We take  $A = C_0^{-\frac{1}{2}}$  and  $\iota = (1-s)/p$  and combining with what we had above we get

$$\sqrt{T} |\langle \mathbb{G}_T, e' \rangle| \lesssim T^{-\frac{s-1}{2p}} \|\mathbb{G}_T\|_{H^s} \|C_0^{-\frac{1}{2}} e\|_{L^2}^{\frac{1-s}{p}} \|\sqrt{T} e\|_{L^2}^{\frac{p+s-1}{p}}.$$

Using Young's inequality for products we have the further bound

$$\sqrt{T} |\langle \mathbb{G}_T, e' \rangle| \lesssim \kappa T^{-\frac{s-1}{p}} \|\mathbb{G}_T\|_{H^s}^2 + \kappa^{-1} \|C_0^{-\frac{1}{2}} e\|_{L^2}^2 + \kappa^{-1} T \|e\|_{L^2}^2.$$

If we combine this with (5.1), choose  $\kappa$  large enough and subtract  $\kappa^{-1} \|C_0^{-\frac{1}{2}} e\|_{L^2}^2 + \kappa^{-1} T \|e\|_{L^2}^2$  from both sides of the inequality we arrive at the bound

$$T\|e\|_{L^2}^2 \leq O_P(1) + T^{-\frac{s-1}{p}} \|\mathbb{G}_T\|_{H^s}^2,$$

which holds on the event  $A_T$ . In view of Theorem 4.1 and since  $s \in (0, 1/2)$  is arbitrary, this completes the proof.  $\square$

Theorem 5.1 only concerns the posterior mean, but we can in fact show that the whole posterior distribution contracts around the true  $b_0$  at the same rate. As usual, we say that the posterior contracts around  $b_0$  at the rate  $\varepsilon_T$  (relative to the  $L^2$ -norm) if for arbitrary positive numbers  $M_T \rightarrow \infty$ ,

$$\mathbb{E}_{b_0} \Pi(b : \|b - b_0\|_{L^2} \geq M_T \varepsilon_T \mid X^T) \rightarrow 0$$

as  $T \rightarrow \infty$ . This essentially says that for large  $T$ , the posterior mass is concentrated in  $L^2$ -balls around  $b_0$  with a radius of the order  $\varepsilon_T$ .

**Theorem 5.2.** *Suppose that  $b_0 \in \dot{H}^p(\mathbb{T})$ . Then for every  $\delta > 0$ , the posterior contracts around  $b_0$  at the rate  $T^{-\frac{p-1/2}{2p}+\delta}$  as  $T \rightarrow \infty$ .*

**Proof.** Set  $\varepsilon_T = T^{-\frac{p-1/2}{2p}+\delta}$ . By the triangle inequality,

$$\begin{aligned} \mathbb{E}_{b_0} \Pi(b : \|b - b_0\|_{L^2} \geq M_T \varepsilon_T \mid X^T) \\ \leq \mathbb{E}_{b_0} \Pi\left(b : \|b - \hat{b}_T\|_{L^2} \geq \frac{M_T \varepsilon_T}{2} \mid X^T\right) + \mathbb{P}_{b_0}\left(\|\hat{b}_T - b_0\|_{L^2} \geq \frac{M_T \varepsilon_T}{2}\right). \end{aligned}$$

By Theorem 5.1 the second term on the right vanishes as  $T \rightarrow \infty$ , hence it suffices to show that  $\Pi(b : \|b - \hat{b}_T\|_{L^2} \geq M_T \varepsilon_T / 2 \mid X^T)$  converges to 0 in  $\mathbb{P}_{b_0}$ -probability. By Markov's inequality, this quantity is bounded by

$$\frac{4}{M_T^2 \varepsilon_T^2} \int \|\hat{b}_T - b\|_{L^2}^2 \Pi(db \mid X^T).$$

Since the integral is equal to the trace of the covariance operator of the centered posterior, it suffices to show that  $\text{tr}(\mathcal{C}_T) = O_P(\varepsilon_T^2)$ .

As before, let  $\lambda_i$  and  $\phi_i$  be the eigenvalues and eigenfunctions of the prior covariance operator  $\mathcal{C}_0$ . For every  $N \in \mathbb{N}$  we have

$$\text{tr}(\mathcal{C}_T) = \sum_{i \leq N} \langle \phi_i, \mathcal{C}_T \phi_i \rangle + \sum_{i > N} \langle \phi_i, \mathcal{C}_T \phi_i \rangle.$$

To bound the second sum on the right we note that in view of (3.3) we have  $\mathcal{C}_T^{-1} \geq \mathcal{C}_0^{-1}$  and it is not difficult to see that as a consequence,  $\mathcal{C}_T \leq \mathcal{C}_0$ . Hence, since  $\lambda_i \sim i^{-2p}$ , the second sum is bounded by a constant times  $N^{1-2p}$ . By Cauchy-Schwarz the first sum is bounded by  $\sum_{i \leq N} \|\mathcal{C}_T \phi_i\|_{L^2}^2$ . To further bound this, we observe that

$$\begin{aligned} \inf_{x \in \mathbb{T}} L_T^\circ(x; X) \|\mathcal{C}_T \phi_i\|_{L^2}^2 &\leq \int_0^1 (\mathcal{C}_T \phi_i(x))^2 L_T^\circ(x; X) dx \\ &\leq \int_0^1 \mathcal{C}_T \phi_i(x) (\mathcal{C}_0^{-1} + L_T^\circ(\cdot; X)) \mathcal{C}_T \phi_i(x) dx \\ &= \int_0^1 \phi_i(x) \mathcal{C}_T \phi_i(x) dx \\ &\leq \|\phi_i\|_{L^2} \|\mathcal{C}_T \phi_i\|_{L^2} = \|\mathcal{C}_T \phi_i\|_{L^2}. \end{aligned}$$

Dividing by  $\|\mathcal{C}_T \phi_i\|_{L^2}$  shows that  $\|\mathcal{C}_T \phi_i\|_{L^2} \leq 1 / \inf_{x \in \mathbb{T}} L_T^\circ(x; X)$  and hence, by the first statement of Theorem 4.1,  $\|\mathcal{C}_T \phi_i\|_{L^2} = O_P(1/T)$ .

Combining what we have we see that  $\text{tr}(\mathcal{C}_T) \leq N O_P(1/T) + N^{1-2p}$  for every  $N \in \mathbb{N}$ . The choice  $N \sim T^{1/(2p)}$  balances the two terms and shows that  $\text{tr}(\mathcal{C}_T) = O_P(T^{(1-2p)/(2p)}) = O_P(\varepsilon_T^2)$ .  $\square$

**Remarks 5.3.** *It is clear from the proof of Theorem 5.2 that the posterior spread  $\int \|\hat{b}_T - b_0\|_{L^2}^2 \Pi(db \mid X^T)$  is always of the order  $T^{(1-2p)/(2p)}$ , regardless of*

the smoothness of the true drift function  $b_0$ . Hence if the rate result of Theorem 5.1 for the posterior mean can be improved, for instance the condition that  $b_0 \in \dot{H}^p(\mathbb{T})$  can be relaxed to the assumption  $b_0 \in \dot{H}^{p-1/2}(\mathbb{T})$  (see the discussion in the introduction), or the  $\delta$  can be removed from the rate, then the result of Theorem 5.2 for the full posterior automatically improves as well.

We also note that the proof of Theorem 5.1 delivers convergence rates in other norms. In particular it yields

$$\|\hat{b}_T - b_0\|_{H^p} = O_{\mathbb{P}_{b_0}}(T^{\frac{1}{4p} + \delta})$$

and hence, by interpolation, we have that the error in the mean converges to zero as

$$\|\hat{b}_T - b_0\|_{H^s} = O_{\mathbb{P}_{b_0}}(T^{\frac{1-2(p-s)}{4p} + \delta})$$

for  $0 \leq s < p - \frac{1}{2}$ .

## 6 Proof of Theorem 4.1

### 6.1 Semimartingale versus diffusion local time

Throughout this section, the drift function  $b \in \dot{C}(\mathbb{T})$  is fixed. The weak solution  $X$  of the SDE (2.1) is a regular diffusion on  $\mathbb{R}$  with scale function  $s$  given by

$$s(x) = \int_{x_0}^x e^{-2 \int_{y_0}^y b(z) dz} dy.$$

We choose  $x_0$  and  $y_0$  such that  $s(0) = 0$  and  $s(1) = 1$ . The speed measure  $m$  has Lebesgue density  $1/s'$ . Since  $b$  is 1-periodic and mean-zero the function  $s'$  is 1-periodic as well. It follows that  $m$  is 1-periodic and that  $s$  satisfies

$$s(x+k) = s(x) + k, \tag{6.1}$$

for all  $x \in \mathbb{R}$  and  $k \in \mathbb{Z}$ .

The periodic local time  $L^\circ$  was defined through the semimartingale local time  $L$  of the diffusion  $X$ , for which we have the occupation times formula (2.3). The diffusion  $X$  also has continuous local time relative to its speed measure, the so-called diffusion local time of  $X$ . We denote this random field by  $(\ell_t(x) : t \geq 0, x \in \mathbb{R})$ . It holds that  $t \mapsto \ell_t(x)$  is continuous and for every  $t \geq 0$  and bounded, measurable function  $f$ ,

$$\int_0^t f(X_u) du = \int_{\mathbb{R}} f(x) \ell_t(x) m(dx) \tag{6.2}$$

(see for instance [20]). For this local time we define a periodic version  $\ell^\circ$  as well, by setting

$$\ell_t^\circ(x) = \sum_{k \in \mathbb{Z}} \ell_t(x+k).$$

The periodicity of  $m$  then implies that for every 1-periodic, bounded, measurable function  $f$ ,

$$\int_0^t f(X_u) du = \int_0^1 f(x) \ell_t^\circ(x) m(dx). \quad (6.3)$$

Comparing this with (2.5) we see that we have the relation  $s'(x)L_T^\circ(x; X) = \ell_T^\circ(x)$  for every  $T \geq 0$  and  $x \in [0, 1]$ . Now note that  $1/s'$  is up to a constant equal to the invariant density  $\rho$  defined by (4.1). Since  $\rho$  is a probability density on  $[0, 1]$  and  $1/s'$  is the density of the speed measure  $m$ , we have  $m[0, 1]\rho = 1/s'$  on  $[0, 1]$ . Therefore, statement (i) of Theorem 4.1 is equivalent to the statement that

$$\sup_{x \in \mathbb{T}} \left| \frac{1}{t} \ell_t^\circ(x) - \frac{1}{m[0, 1]} \right| \rightarrow 0 \quad (6.4)$$

a.s. as  $t \rightarrow \infty$ , and statement (ii) is equivalent to the asymptotic tightness of

$$x \mapsto \sqrt{t} \left( \frac{1}{t} \ell_t^\circ(x) - \frac{1}{m[0, 1]} \right) \quad (6.5)$$

in  $H^\alpha(\mathbb{T})$  for every  $\alpha \in [0, 1/2)$ . We will prove these statements in the subsequent sections.

## 6.2 A representation of the local time up to winding times

If we do not write it explicitly, we work under the measure  $\mathbb{P}_0$ , i.e.  $X$  is started in 0. We define a sequence of  $\mathbb{P}_0$ -a.s. finite stopping times  $\tau_0, \tau_1, \dots$  by setting  $\tau_0 = 0$ ,  $\tau_1$  is the first time  $X$  exits  $[-1, 1]$ ,  $\tau_2$  is the first time after  $\tau_1$  that  $X$  exits  $[X_{\tau_1} - 1, X_{\tau_1} + 1]$ , etc. (Note that if we define a process  $Z$  on the complex unit circle by  $Z_t = \exp(2i\pi X_t)$ , then  $\tau_k$  is the time that the process  $Z$  completes its  $k$ th winding of the circle.)

The following theorem gives a representation for the periodic local time of  $X$  up till the  $n$ th winding time. The representation involves a stochastic integral relative to  $s(X)$ . The process  $s(X)$  is a diffusion in natural scale, hence a time-changed Brownian motion, and hence a continuous local martingale.

**Theorem 6.1.** *For  $x \in (0, 1)$ ,*

$$\frac{1}{n} \ell_{\tau_n}^\circ(x) - 1 = \frac{1}{n} \sum_{k=1}^n U_k(x),$$

where  $U_1, \dots, U_n$  are i.i.d. continuous random functions, distributed as

$$U(x) = \ell_{\tau_1}(x) + \ell_{\tau_1}(x - 1) - 1 \quad (6.6)$$

$$= X_{\tau_1}(1 - 2s(x)) + 2 \int_0^{\tau_1} \phi_x(X_u) ds(X_u), \quad (6.7)$$

where  $\phi_x = 1_{(x-1, \infty)} - 1_{(-\infty, x]}$ .

**Proof.** For  $k \in \mathbb{N}$  we write  $X^k = (X_{\tau_{k-1}+t} - X_{\tau_{k-1}} : t \geq 0)$  and  $\tau_1^k = \inf\{t : |X_t^k| = 1\}$ . By Lemma 6.2 ahead, the processes  $(X_t^k : t \in [0, \tau_1^k])$  are independent and have the same distribution as  $(X_t : t \in [0, \tau_1])$ . It follows that for  $x \in (0, 1)$ , with  $\ell^Z$  denoting the diffusion local time of the diffusion  $Z$ ,

$$\frac{1}{n} \ell_{\tau_n}^{\circ}(x) - 1 = \frac{1}{n} \sum_{k=1}^n U_k(x), \quad (6.8)$$

where

$$U_k(x) = \ell_{\tau_1^k}^{X^k}(x) + \ell_{\tau_1^k}^{X^k}(x-1) - 1,$$

and the  $U_k$  are independent copies of the random function  $U$  defined by (6.6).

Now let  $Y = s(X)$ . Then  $Y$  is a regular diffusion in natural scale (i.e. the identity function is its scale function) and the speed measure  $m^Y$  of  $Y$  is related to the speed measure  $m$  of  $X$  by  $m = m^Y \circ s$ . It is easily seen that for  $\ell^Y$  the local time of  $Y$  relative to its speed measure  $m^Y$ , we have  $\ell_t(x) = \ell_t^Y(s(x))$ . For diffusions in natural scale, the diffusion local time coincides with the semimartingale local time (see [29], Section V.49). In particular, the Tanaka-Méyer formula holds:

$$\ell_t^Y(x) = |Y_t - x| - |x| - \int_0^t \text{sign}(Y_u - x) dY_u \quad (6.9)$$

under  $\mathbb{P}_0$ . In view of (6.1)  $\tau_1$  is also the first time that  $Y$  exits  $[-1, 1]$ , so we have that  $X_{\tau_1} = Y_{\tau_1}$ . Using also the fact that the scale function  $s$  is strictly increasing, we obtain

$$\ell_{\tau_1}(x) = |X_{\tau_1} - s(x)| - |s(x)| - \int_0^{\tau_1} \text{sign}(X_u - x) ds(X_u). \quad (6.10)$$

Together with (6.1) this implies that (6.7) holds.  $\square$

The proof of the theorem uses the following lemma, which implies that  $X$  “starts afresh” after every winding time  $\tau_k$ . Let  $(\mathcal{F}_t : t \geq 0)$  denote the natural filtration of the process  $X$ .

**Lemma 6.2.** *For every  $\mathbb{P}_0$ -a.s. finite stopping time  $\tau$  such that  $X_\tau \in \mathbb{Z}$  a.s., it holds that the process  $(X_{\tau+t} - X_\tau : t \geq 0)$  is independent of  $\mathcal{F}_\tau$  and has the same law as  $X$  under  $\mathbb{P}_0$ .*

**Proof.** Fix a measurable subset  $C \subset C[0, \infty)$ . By the strong Markov property we have

$$\mathbb{P}_0(X_{\tau+} - X_\tau \in C | \mathcal{F}_\tau) = f(X_\tau)$$

a.s., where  $f(x) = \mathbb{P}_x(X - X_0 \in C)$ . The periodicity of the drift function implies that for every  $k \in \mathbb{Z}$ ,  $f(k) = \mathbb{P}_k(X - k \in C) = \mathbb{P}_0(X \in C)$ . Hence we have

$$\mathbb{P}_0(X_{\tau+} - X_\tau \in C | \mathcal{F}_\tau) = \mathbb{P}_0(X \in C),$$

a.s., which completes the proof.  $\square$

Since we will be interested in the local time up till a deterministic time  $t$ , it is necessary to deal with the time interval between  $t$  and the previous or next winding time. The following lemma will be used for that. For  $t \geq 0$ , let the  $\mathbb{Z}_+$ -valued random variable  $n_t$  be such that  $\tau_{n_t}$  is the last winding time less or equal to  $t$ , so  $\tau_{n_t} \leq t < \tau_{n_t+1}$ .

**Lemma 6.3.** *For all  $t \geq 0$  and Borel sets  $B \subset C[0, 1]$ ,*

$$\mathbb{P}_0(\ell_{\tau_{n_t}+1}^\circ - \ell_t^\circ \in B) = \mathbb{E}_0 \mathbb{P}_{X_t - X_{n_t}}(\ell_{\tau_1}^\circ \in B).$$

**Proof.** We split up the event of interest according to the position of  $X$  at time  $\tau_{n_t}$ . For  $k \in \mathbb{Z}$  we have

$$\mathbb{P}_0(\ell_{\tau_{n_t}+1}^\circ - \ell_t^\circ \in B; X_{\tau_{n_t}} = k) = \mathbb{P}_0(\ell_{\sigma_{t,k}}^\circ - \ell_t^\circ \in B; X_{\tau_{n_t}} = k),$$

where  $\sigma_{t,k} = \inf\{s > t : |X_s - k| \geq 1\}$ . Let  $(\mathcal{F}_s : s \geq 0)$  be the natural filtration of the process  $X$ . Since  $X_{\tau_{n_t}}$  is  $\mathcal{F}_t$ -measurable, conditioning on  $\mathcal{F}_t$  gives

$$\mathbb{P}_0(\ell_{\tau_{n_t}+1}^\circ - \ell_t^\circ \in B; X_{\tau_{n_t}} = k) = \mathbb{E}_0 1_{\{X_{\tau_{n_t}}=k\}} \mathbb{P}_0(\ell_{\sigma_{t,k}}^\circ - \ell_t^\circ \in B | \mathcal{F}_t).$$

By the Markov property, the conditional probability equals  $\mathbb{P}_{X_t}(\ell_{\sigma_{0,k}}^\circ \in B)$ . By the periodicity of the drift function, this is equal to  $\mathbb{P}_{X_t-k}(\ell_{\sigma_{0,0}}^\circ \in B)$ . Since  $\sigma_{0,0} = \tau_1$ , we obtain

$$\mathbb{P}_0(\ell_{\tau_{n_t}+1}^\circ - \ell_t^\circ \in B; X_{\tau_{n_t}} = k) = \mathbb{E}_0 1_{\{X_{\tau_{n_t}}=k\}} \mathbb{P}_{X_t-k}(\ell_{\tau_1}^\circ \in B).$$

Summation over  $k$  completes the proof.  $\square$

### 6.3 Proof of statement (i) of Theorem 4.1

In this section we prove that (6.4) holds a.s. for  $T \rightarrow \infty$ , which is equivalent to statement (i) of Theorem 4.1.

According to Theorem 6.1 we have

$$\frac{1}{n} \ell_{\tau_n}^\circ(x) - 1 = \frac{1}{n} \sum_{k=1}^n U_k(x),$$

where the  $U_k$  are independent copies of the continuous random function on  $[0, 1]$  given by (6.6). Now  $\mathbb{E}\|U\|_\infty \leq 1 + 2\mathbb{E} \sup_{|x| \leq 1} \ell_{\tau_1}(x)$ . To bound the expectation, we again use the fact that  $\ell_{\tau_1}(x) = \ell_{\tau_1}^Y(s(x))$ , for  $Y = s(X)$ . Relation (6.1) implies that  $\sup_{|x| \leq 1} \ell_{\tau_1}(x) = \sup_{|x| \leq 1} \ell_{\tau_1}^Y(x)$ . Applying the BDG-type inequality for local times to the stopped continuous local martingale  $Y^{\tau_1}$  (see [27], Theorem XI.(2.4)) we then see that for some constant  $C > 0$ ,

$$\mathbb{E}\|U\|_\infty \leq 1 + 2\mathbb{E} \sup_{|x| \leq 1} \ell_{\tau_1}^Y(x) \leq 1 + C\mathbb{E} \sup_{t \leq \tau_1} |Y_t| < \infty.$$

Since by (6.1) it holds that  $X_{\tau_1} = \pm 1$  with equal probability, it easily derived from (6.7) that  $\mathbb{E}U(x) = 0$ . By the Banach space version of Kolmogorov's law of large numbers (see [23], Corollary 7.10), it follows that

$$\sup_{x \in [0,1]} \left| \frac{1}{n} \ell_{\tau_n}^{\circ}(x) - 1 \right| \rightarrow 0 \quad (6.11)$$

a.s..

The random variables  $\tau_1, \tau_2 - \tau_1, \tau_3 - \tau_2, \dots$  are i.i.d., so by the law of large numbers,  $\tau_n/n \rightarrow \mathbb{E}\tau_1$  a.s.. Applying relation (6.2) with  $t = \tau_1$  and  $f \equiv 1$  we see that

$$\mathbb{E}\tau_1 = \int_{-1}^1 \mathbb{E}\ell_{\tau_1}(x) m(dx).$$

Since  $X_{\tau_1} = \pm 1$  with equal probability, (6.10) implies that  $\mathbb{E}\ell_{\tau_1}(x) = 1 - |s(x)|$ . Using (6.1) and the periodicity of  $m$ , it follows that

$$\begin{aligned} \mathbb{E}\tau_1 &= \int_{-1}^0 (1 + s(x)) m(dx) + \int_0^1 (1 - s(x)) m(dx) \\ &= \int_0^1 (1 + s(x-1)) m(dx) + \int_0^1 (1 - s(x)) m(dx) = m[0, 1]. \end{aligned}$$

Combining (6.11) with the fact that  $\tau_n/n \rightarrow m[0, 1]$  a.s., we find that

$$\sup_{x \in [0,1]} \left| \frac{1}{\tau_n} \ell_{\tau_n}^{\circ}(x) - \frac{1}{m[0, 1]} \right| \rightarrow 0 \quad (6.12)$$

a.s..

Now let  $n_t$  be defined as before Lemma 6.3, so that  $\tau_{n_t} \leq t < \tau_{n_t+1}$ . Then as  $t \rightarrow \infty$  it holds that  $n_t \rightarrow \infty$  and hence  $\tau_{n_t}/n_t \rightarrow m[0, 1]$  a.s. and  $\tau_{n_t+1}/n_t \rightarrow m[0, 1]$  a.s.. It follows that  $n_t/t \rightarrow 1/m[0, 1]$  a.s., and therefore also  $\tau_{n_t}/t \rightarrow 1$  a.s.. We can write

$$\frac{1}{t} \ell_t^{\circ}(x) = \frac{\tau_{n_t}}{t} \frac{1}{\tau_{n_t}} \ell_{\tau_{n_t}}^{\circ}(x) + \frac{1}{t} (\ell_t^{\circ}(x) - \ell_{\tau_{n_t}}^{\circ}(x)).$$

Relation (6.12) shows that a.s., the first term on the right converges uniformly to  $1/m[0, 1]$ . The second term is nonnegative and bounded by

$$\frac{1}{t} (\ell_{\tau_{n_t}+1}^{\circ}(x) - \ell_{\tau_{n_t}}^{\circ}(x)) = \frac{\tau_{n_t+1}}{t} \frac{1}{\tau_{n_t+1}} \ell_{\tau_{n_t+1}}^{\circ}(x) - \frac{\tau_{n_t}}{t} \frac{1}{\tau_{n_t}} \ell_{\tau_{n_t}}^{\circ}(x),$$

which converges uniformly to 0 by the preceding. This completes the proof of (6.4) and hence of statement (i) of Theorem 4.1.

## 6.4 Proof of statement (ii) of Theorem 4.1

In this final subsection we prove that the random maps (6.5) are asymptotically tight in the space in  $H^{\alpha}(\mathbb{T})$  for every  $\alpha \in [0, 1/2)$ , which is equivalent to statement (ii) of Theorem 4.1. It is most convenient and of course not restrictive to

work with the complex Sobolev spaces. Let  $e_k(x) = \exp(i2k\pi x)$ ,  $k \in \mathbb{Z}$ , be the standard complex exponential basis of  $L^2[0, 1]$ . For  $\alpha \geq 0$ , define the associated Sobolev space

$$H^\alpha[0, 1] = \left\{ f \in L^2[0, 1] : \|f\|_{H^\alpha}^2 = \sum |k|^{2\alpha} |\langle f, e_k \rangle|^2 < \infty \right\},$$

where  $\langle f, g \rangle = \int_0^1 f(x)\bar{g}(x) dx$  is the usual inner product on  $L^2[0, 1]$ .

By the representation of the local time given by Theorem 6.1 and the central limit theorem for Hilbert space-valued random elements (e.g. [23], Corollary 10.9), we have that

$$\sqrt{n} \left( \frac{1}{n} \ell_{\tau_n}^\circ - 1 \right) \quad (6.13)$$

converges weakly in  $H^\alpha[0, 1]$  if

1.  $\mathbb{E}\|U\|_{H^\alpha}^2 < \infty$ .
2.  $\mathbb{E}U = 0$  (where the expectation is to be interpreted as a Pettis integral),

We will show that these conditions hold if (and only if)  $\alpha < 1/2$ . Slightly abusing notation, denote the two functions on the right of (6.7) by  $U_1$  and  $U_2$ . We will show that conditions 1–2 hold for  $U_1$  and  $U_2$  separately.

To show that the conditions hold for  $U_1$ , recall that  $X_{\tau_1} \pm 1$  with equal probability. Hence,  $\mathbb{E}U_1 = 0$  and  $\mathbb{E}\|U_1\|_{H^\alpha}^2 = \|1 - 2s\|_{H^\alpha}^2 < \infty$ .

As for  $U_2$ , using (6.7) and the stochastic Fubini theorem it is readily checked that

$$\langle U_2, e_k \rangle = 2 \int_0^{\tau_1} c_k(X_u) ds(X_u),$$

where

$$c_k(x) = \begin{cases} \frac{1 - e^{i2k\pi}}{i2k\pi} & \text{if } x + 1 \leq 0, \\ \frac{e^{i2k(x+1)\pi} - e^{i2k\pi}}{i2k\pi} & \text{if } x \leq 0 \leq x + 1 \leq 1, \\ \frac{e^{i2k\pi x} - 1}{i2k\pi} & \text{if } 0 \leq x \leq 1 \leq x + 1, \\ \frac{e^{i2k\pi} - 1}{i2k\pi} & \text{if } x \geq 1. \end{cases}$$

To show that condition 1. holds for  $U_2$ , note that for  $u \leq \tau_1$  it holds that  $|X_u| \leq 1$ . It is straightforward to see that for  $|x| \leq 1$ , we have  $|c_k(x)| \leq C(1 + |k|)^{-1}$  for some  $C > 0$ . Therefore, by the Itô isometry,

$$\begin{aligned} \mathbb{E} \sum |k|^{2\alpha} \left| \int_0^{\tau_1} c_k(X_u) ds(X_u) \right|^2 &= \sum |k|^{2\alpha} \mathbb{E} \int_0^{\tau_1} |c_k(X_u)|^2 d\langle s(X) \rangle_u \\ &\leq C^2 \mathbb{E} \langle s(X) \rangle_{\tau_1} \sum \frac{|k|^{2\alpha}}{(1 + |k|)^2}. \end{aligned}$$



The sum on the right is finite if  $\alpha \in [0, 1/2)$ . For the diffusion  $Y = s(X)$  the diffusion local time coincides with the semimartingale local time, hence

$$\mathbb{E} \langle s(X) \rangle_{\tau_1} = \mathbb{E} \int \ell_{\tau_1}^Y(x) dx = \int_{-1}^1 \mathbb{E} \ell_{\tau_1}^Y(x) dx.$$

The Tanaka-Méyer formula and optional stopping imply that for  $|x| \leq 1$ ,

$$\mathbb{E} \ell_{\tau_1}^Y(x) = \mathbb{E} |Y_{\tau_1} - x| - |x| = 1 - |x|.$$

Hence,  $U_2$  satisfies condition 1. Finally, note that to show that  $\mathbb{E} U_2 = 0$ , it suffices to show that  $\mathbb{E} U_2(x) = 0$  for every fixed  $x \in (0, 1)$ . But this follows readily from (6.6) again, by optional stopping. So indeed the random maps (6.13) converge in  $H^\alpha[0, 1]$  for every  $\alpha \in [0, 1/2)$ .

To complete the proof we consider the decomposition

$$\begin{aligned} & \sqrt{t} \left( \frac{\ell_t^\circ}{t} - \frac{1}{m[0, 1]} \right) \\ &= \sqrt{n_t + 1} \left( \frac{\ell_{\tau_{n_t+1}}^\circ}{n_t + 1} - 1 \right) \sqrt{\frac{n_t + 1}{t}} + \sqrt{t} \left( \frac{n_t + 1}{t} - \frac{1}{m[0, 1]} \right) - \frac{\ell_{\tau_{n_t+1}}^\circ - \ell_t^\circ}{\sqrt{t}}. \end{aligned}$$

Since  $n_t/t \rightarrow 1/m[0, 1]$  a.s., the tightness of the maps (6.13) implies that the first term is asymptotically tight. By the central limit theorem,  $\sqrt{n}(\tau_n/n - m[0, 1])$  converges in distribution. Together with the inequality  $\tau_{n_t} \leq t < \tau_{n_t+1}$  and the delta method this implies that the second term is asymptotically tight as well. For the last term, note that by Lemma 6.3 we have, for  $M > 0$ ,

$$\mathbb{P}_0 \left( \left\| \frac{\ell_{\tau_{n_t+1}}^\circ - \ell_t^\circ}{\sqrt{t}} \right\|_{H^\alpha} > M \right) \leq \sup_{|a| \leq 1} \mathbb{P}_a(\|\ell_{\tau_1}\|_{H^\alpha} > M\sqrt{t}) \leq \frac{1}{M^2 t} \sup_{|a| \leq 1} \mathbb{E}_a \|\ell_{\tau_1}\|_{H^\alpha}^2.$$

Similar considerations as used to show that condition 1. above holds for  $U_2$  show that the supremum over  $a$  on the right-hand side is bounded. We conclude that the last term in the decomposition is  $o_P(1)$ . This completes the proof.

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